

一元八次方程求根公式 (天珩公式)

標準型：(係數 $\in\mathbb{R}$  且  $a\neq 0$ )

$$ax^8+bx^7+cx^6+dx^5+ex^4+fx^3+gx^2+hx+j=0$$

重根判別式：

$$\begin{cases} G=7b^2-16ac \\ H=7b^3-24abc+32a^2d \\ J=15b^2c-100abd+256a^2e \\ K=b^3c-8ab^2d+32a^2be-64a^3f \\ L=5b^3d-72ab^2e+528a^2bf-2048a^3g \\ M=b^4d-16ab^3e+144a^2b^2f-896a^3bg+3584a^4h \\ N=b^4e-28ab^3f+464a^2b^2g-5952a^3bh+65536a^4j \end{cases}$$

$$\begin{cases} D=3G^2-10bH-8aJ \\ E=G^3+bGH-4aGJ+16a^2L \\ F=3G^4-2b^5H+16a^2J^2-16aG^2J+62ab^3K+32a^2GL-156a^2bM-64a^3N \end{cases}$$

$$\begin{cases} A=D^2-3F \\ B=DF-9E^2 \\ C=F^2-3DE^2 \end{cases}$$

$$(*) \begin{cases} P=GH+b^2H-16aK \\ Q=E+8H^2-4b^3H+44abK \\ R=14G^2H+105bH^2+3b^4H-56aHJ-96ab^2K+256a^2M \\ S=8EG-4F-9bDH+bG^2H \end{cases}$$

總判別式： $\Delta=B^2-4AC$

①當  $G=H=J=K=L=M=N=0$  時，方程有一個八重實根。

$$\text{公式 1: } x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = x_7 = x_8 = -\frac{b}{8a} = -\frac{2c}{7b} = -\frac{d}{2c} = -\frac{4e}{5d} = -\frac{5f}{4e} = -\frac{2g}{f} = -\frac{7h}{2g} = -\frac{8j}{h}$$

②當  $H=K=M=D=E=F=0$  時，方程有兩個四重根。

(1)若  $7b^2 > 16ac$ ，方程有八個實根。

$$\text{公式 2: } x_{1,2} = x_{3,4} = x_{5,6} = x_{7,8} = \frac{-b \pm \sqrt{7b^2 - 16ac}}{8a}$$

(2)若  $7b^2 < 16ac$ ，方程有八個虛根。

$$\text{公式 3: } x_{1,2} = x_{3,4} = x_{5,6} = x_{7,8} = -\frac{b}{8a} \pm \frac{\sqrt{16ac - 7b^2}}{8a} i$$

③當  $DEF \neq 0$ ， $H=K=M=A=B=C=0$  時，方程有兩個三重根。

(1)若  $D^2G < \min\{-9DE, 3DE\}$ ，方程有八個虛根。

$$\text{公式 4: } x_{1,2} = -\frac{b}{8a} \pm \frac{\sqrt{-D^2G - 9DE}}{8aD} i \quad x_{3,4} = x_{5,6} = x_{7,8} = -\frac{b}{8a} \pm \frac{\sqrt{-D^2G + 3DE}}{8aD} i$$

(2)若  $-9DE \leq D^2G < 3DE$ ，方程有兩個實根和六個虛根。

$$\text{公式 5: } x_{1,2} = \frac{-bD \pm \sqrt{D^2G + 9DE}}{8aD} \quad x_{3,4} = x_{5,6} = x_{7,8} = -\frac{b}{8a} \pm \frac{\sqrt{-D^2G + 3DE}}{8aD} i$$

(3) 若  $3DE \leq D^2G < -9DE$ , 方程有六個實根和兩個虛根。

$$\text{公式 6: } x_{1,2} = -\frac{b}{8a} \pm \frac{\sqrt{-D^2G - 9DE}}{8aD} i \quad x_{3,4} = x_{5,6} = x_{7,8} = \frac{-bD \pm \sqrt{D^2G - 3DE}}{8aD}$$

(4) 若  $D^2G \geq \max\{-9DE, 3DE\}$ , 方程有八個實根。

$$\text{公式 7: } x_{1,2} = \frac{-bD \pm \sqrt{D^2G + 9DE}}{8aD} \quad x_{3,4} = x_{5,6} = x_{7,8} = \frac{-bD \pm \sqrt{D^2G - 3DE}}{8aD}$$

④ 當  $D \neq 0, H=K=M=E=F=0$  時, 方程有四個二重根。

(1) 若  $D > 0$  且  $G < -\sqrt{D}$ , 方程有八個虛根。

$$\text{公式 8: } x_{1,2} = x_{3,4} = -\frac{b}{8a} \pm \frac{\sqrt{-G + \sqrt{D}}}{8a} i \quad x_{5,6} = x_{7,8} = -\frac{b}{8a} \pm \frac{\sqrt{-G - \sqrt{D}}}{8a} i$$

(2) 若  $D > 0$  且  $-\sqrt{D} \leq G < \sqrt{D}$ , 方程有四個實根和四個虛根。

$$\text{公式 9: } x_{1,2} = x_{3,4} = \frac{-b \pm \sqrt{G + \sqrt{D}}}{8a} \quad x_{5,6} = x_{7,8} = -\frac{b}{8a} \pm \frac{\sqrt{-G + \sqrt{D}}}{8a} i$$

(3) 若  $D > 0$  且  $G \geq \sqrt{D}$ , 方程有八個實根。

$$\text{公式 10: } x_{1,2} = x_{3,4} = \frac{-b \pm \sqrt{G + \sqrt{D}}}{8a} \quad x_{5,6} = x_{7,8} = \frac{-b \pm \sqrt{G - \sqrt{D}}}{8a}$$

(4) 若  $D < 0$ , 方程有八個虛根。

$$\text{公式 11: } x_{1,2} = x_{3,4} = \frac{-2b + \sqrt{2G + 2\sqrt{G^2 - D}}}{16a} \pm \frac{\sqrt{-2G + 2\sqrt{G^2 - D}}}{16a} i$$

$$x_{5,6} = x_{7,8} = \frac{-2b - \sqrt{2G + 2\sqrt{G^2 - D}}}{16a} \pm \frac{\sqrt{-2G + 2\sqrt{G^2 - D}}}{16a} i$$

⑤ 當  $ABC \neq 0, H=K=M=\Delta=0$  時, 方程有兩個二重根。

(1) 若  $AB > 0$ 。

$$\text{公式 12: } x_{1,2} = \frac{-b \pm \sqrt{G + 2AE/B + \sqrt{2B/A}}}{8a} \quad x_{3,4} = \frac{-b \pm \sqrt{G + 2AE/B - \sqrt{2B/A}}}{8a}$$

$$x_{5,6} = x_{7,8} = \frac{-b \pm \sqrt{G - 2AE/B}}{8a}$$

(2) 若  $AB < 0$ , 方程有至少四個虛根。

$$\text{公式 13: } x_{1,2} = \frac{-2b + \sqrt{2G + 4AE/B + 2\sqrt{G^2 + D + 4AEG/B - 3B/A}}}{16a} \pm \frac{\sqrt{-2G - 4AE/B + 2\sqrt{G^2 + D + 4AEG/B - 3B/A}}}{16a} i$$

$$x_{3,4} = \frac{-2b - \sqrt{2G + 4AE/B + 2\sqrt{G^2 + D + 4AEG/B - 3B/A}}}{16a} \pm \frac{\sqrt{-2G - 4AE/B + 2\sqrt{G^2 + D + 4AEG/B - 3B/A}}}{16a} i$$

$$x_{5,6} = x_{7,8} = \frac{-b \pm \sqrt{G - 2AE/B}}{8a}$$

⑥ 當  $\Delta > 0, H=K=M=0$  時, 方程有至少四個虛根。

$$z_{1,2} = AD + 3 \cdot \frac{-B \pm \sqrt{B^2 - 4AC}}{2} \quad z = D^2 - D(\sqrt[3]{z_1} + \sqrt[3]{z_2}) + (\sqrt[3]{z_1} + \sqrt[3]{z_2})^2 - 3A$$

$$y = \frac{4a \pm \sqrt[3]{z_1} + \sqrt[3]{z_2} + \sqrt{z} - G \operatorname{sgn}(E) \sqrt{3D + 3(\sqrt[3]{z_1} + \sqrt[3]{z_2})}}{6} \quad y_{1,2} = \sqrt{y} \pm \frac{3G - \operatorname{sgn}(E) \sqrt{3D + 3(\sqrt[3]{z_1} + \sqrt[3]{z_2})}}{6}$$

$$\text{公式 14: } x_{1,2} = \frac{-b \pm \sqrt{G + \operatorname{sgn}(E) \sqrt{(D + \sqrt[3]{z_1} + \sqrt[3]{z_2})/3} + \sqrt{(2D - \sqrt[3]{z_1} - \sqrt[3]{z_2} + 2\sqrt{z})/3}}}{8a}$$

$$x_{3,4} = \frac{-b \pm \sqrt{G + \operatorname{sgn}(E) \sqrt{(D + \sqrt[3]{z_1} + \sqrt[3]{z_2})/3} - \sqrt{(2D - \sqrt[3]{z_1} - \sqrt[3]{z_2} + 2\sqrt{z})/3}}}{8a}$$

$$x_{5,6} = \frac{-b + \sqrt{y_1}}{8a} \pm \frac{\sqrt{y_2}}{8a} i \quad x_{7,8} = \frac{-b - \sqrt{y_1}}{8a} \pm \frac{\sqrt{y_2}}{8a} i$$

⑦當 $\Delta < 0$ ,  $H=K=M=0$ 時。

$$\theta = \arccos \frac{3B - 2AD}{2A\sqrt{A}} \quad y_{1,2} = \frac{D + \sqrt{A}(\cos \frac{\theta}{3} \pm \sqrt{3} \sin \frac{\theta}{3})}{3} \quad y_3 = \frac{D - 2\sqrt{A} \cos \frac{\theta}{3}}{3}$$

(1)若 $E=0$ 且 $D>0$ 且 $F>0$ 。

$$\text{公式 15: } x_{1,2} = \frac{-b \pm \sqrt{G + \sqrt{D + 2\sqrt{F}}}}{8a} \quad x_{3,4} = \frac{-b \pm \sqrt{G + \sqrt{D - 2\sqrt{F}}}}{8a}$$

$$x_{5,6} = \frac{-b \pm \sqrt{G - \sqrt{D + 2\sqrt{F}}}}{8a} \quad x_{7,8} = \frac{-b \pm \sqrt{G - \sqrt{D - 2\sqrt{F}}}}{8a}$$

(2)若 $E=0$ 且 $D<0$ 且 $F>0$ ，方程有八個虛根。

$$\text{公式 16: } x_{1,2} = \frac{-2b + \sqrt{2G + 2\sqrt{G^2 - D + 2\sqrt{F}}} \pm \sqrt{-2G + 2\sqrt{G^2 - D + 2\sqrt{F}}}}{16a} i$$

$$x_{3,4} = \frac{-2b + \sqrt{2G + 2\sqrt{G^2 - D - 2\sqrt{F}}} \pm \sqrt{-2G + 2\sqrt{G^2 - D - 2\sqrt{F}}}}{16a} i$$

$$x_{5,6} = \frac{-2b - \sqrt{2G + 2\sqrt{G^2 - D + 2\sqrt{F}}} \pm \sqrt{-2G + 2\sqrt{G^2 - D + 2\sqrt{F}}}}{16a} i$$

$$x_{7,8} = \frac{-2b - \sqrt{2G + 2\sqrt{G^2 - D - 2\sqrt{F}}} \pm \sqrt{-2G + 2\sqrt{G^2 - D - 2\sqrt{F}}}}{16a} i$$

(3)若 $E=0$ 且 $F<0$ ，方程有八個虛根。

$$\text{公式 17: } x_{1,2} = \frac{-2b + \sqrt{2G + \sqrt{2D + 2\sqrt{A-F}} + 2\sqrt{G^2 + G\sqrt{2D + 2\sqrt{A-F}} + \sqrt{A-F}}} \pm \sqrt{-2G - \sqrt{2D + 2\sqrt{A-F}} + 2\sqrt{G^2 + G\sqrt{2D + 2\sqrt{A-F}} + \sqrt{A-F}}}}{16a} i$$

$$x_{3,4} = \frac{-2b - \sqrt{2G + \sqrt{2D + 2\sqrt{A-F}} + 2\sqrt{G^2 + G\sqrt{2D + 2\sqrt{A-F}} + \sqrt{A-F}}} \pm \sqrt{-2G - \sqrt{2D + 2\sqrt{A-F}} + 2\sqrt{G^2 + G\sqrt{2D + 2\sqrt{A-F}} + \sqrt{A-F}}}}{16a} i$$

$$x_{5,6} = \frac{-2b + \sqrt{2G + \sqrt{2D + 2\sqrt{A-F}} + 2\sqrt{G^2 + G\sqrt{2D + 2\sqrt{A-F}} + \sqrt{A-F}}} \pm \sqrt{-2G - \sqrt{2D + 2\sqrt{A-F}} + 2\sqrt{G^2 + G\sqrt{2D + 2\sqrt{A-F}} + \sqrt{A-F}}}}{16a}$$

$$\pm \frac{\sqrt{-2G - \sqrt{2D + 2\sqrt{A-F}} + 2\sqrt{G^2 + G\sqrt{2D + 2\sqrt{A-F}} + \sqrt{A-F}}}}{16a} i$$

$$X_{5,6} = \frac{-2b + \sqrt{2G - \sqrt{2D + 2\sqrt{A-F}} + 2\sqrt{G^2 - G\sqrt{2D + 2\sqrt{A-F}} + \sqrt{A-F}}}}{16a}$$

$$\pm \frac{\sqrt{-2G + \sqrt{2D + 2\sqrt{A-F}} + 2\sqrt{G^2 - G\sqrt{2D + 2\sqrt{A-F}} + \sqrt{A-F}}}}{16a} i$$

$$X_{7,8} = \frac{-2b - \sqrt{2G - \sqrt{2D + 2\sqrt{A-F}} + 2\sqrt{G^2 - G\sqrt{2D + 2\sqrt{A-F}} + \sqrt{A-F}}}}{16a}$$

$$\pm \frac{\sqrt{-2G + \sqrt{2D + 2\sqrt{A-F}} + 2\sqrt{G^2 - G\sqrt{2D + 2\sqrt{A-F}} + \sqrt{A-F}}}}{16a} i$$

(4) 若  $E \neq 0$  且  $D > 0$  且  $F > 0$ 。

$$\text{公式 18: } x_{1,2} = \frac{-b \pm \sqrt{G + \sqrt{y_1} + \sqrt{y_2} + \text{sgn}(E)\sqrt{y_3}}}{8a} \quad x_{3,4} = \frac{-b \pm \sqrt{G - \sqrt{y_1} - \sqrt{y_2} + \text{sgn}(E)\sqrt{y_3}}}{8a}$$

$$x_{5,6} = \frac{-b \pm \sqrt{G + \sqrt{y_1} - \sqrt{y_2} - \text{sgn}(E)\sqrt{y_3}}}{8a} \quad x_{7,8} = \frac{-b \pm \sqrt{G - \sqrt{y_1} + \sqrt{y_2} - \text{sgn}(E)\sqrt{y_3}}}{8a}$$

(5) 若  $E \neq 0$  且  $\min\{D, F\} \leq 0$ ，方程有八個虛根。

$$\text{公式 19: } x_{1,2} = \frac{-2b + \sqrt{2G + 2\sqrt{y_1} + 2\sqrt{G^2 + 2G\sqrt{y_1} + 2y_1 - D - 2E/\sqrt{y_1}}}}{16a} \pm \frac{\sqrt{-2G - 2\sqrt{y_1} + 2\sqrt{G^2 + 2G\sqrt{y_1} + 2y_1 - D - 2E/\sqrt{y_1}}}}{16a} i$$

$$x_{3,4} = \frac{-2b - \sqrt{2G + 2\sqrt{y_1} + 2\sqrt{G^2 + 2G\sqrt{y_1} + 2y_1 - D - 2E/\sqrt{y_1}}}}{16a} \pm \frac{\sqrt{-2G - 2\sqrt{y_1} + 2\sqrt{G^2 + 2G\sqrt{y_1} + 2y_1 - D - 2E/\sqrt{y_1}}}}{16a} i$$

$$x_{5,6} = \frac{-2b + \sqrt{2G - 2\sqrt{y_1} + 2\sqrt{G^2 - 2G\sqrt{y_1} + 2y_1 - D + 2E/\sqrt{y_1}}}}{16a} \pm \frac{\sqrt{-2G + 2\sqrt{y_1} + 2\sqrt{G^2 - 2G\sqrt{y_1} + 2y_1 - D + 2E/\sqrt{y_1}}}}{16a} i$$

$$x_{7,8} = \frac{-2b - \sqrt{2G - 2\sqrt{y_1} + 2\sqrt{G^2 - 2G\sqrt{y_1} + 2y_1 - D + 2E/\sqrt{y_1}}}}{16a} \pm \frac{\sqrt{-2G + 2\sqrt{y_1} + 2\sqrt{G^2 - 2G\sqrt{y_1} + 2y_1 - D + 2E/\sqrt{y_1}}}}{16a} i$$

⑧ 當  $H \neq 0$ ， $P=Q=R=0$ ， $S > 0$  時，方程有八個虛根。

$$y_1 = 5G^2 + 45bH - 24aJ \quad z_1 = 19G^3 + 135bGH - 108H^2 - 72aGJ$$

$$y_2 = -z_1 + \sqrt{\frac{y_1^3 + z_1^2 - 27S(y_1 + 3G^2) + \sqrt{(y_1^3 + z_1^2 - 27S(y_1 + 3G^2))^2 + 81S(y_1^2 + 2Gz_1 - 3S)^2}}{2}}$$

$$z_2 = -9G\sqrt{S} + \operatorname{sgn}(y_1^2 + 2Gz_1 - 3S) \sqrt{\frac{27S(y_1 + 3G^2) - y_1^3 - z_1^2 + \sqrt{(y_1^3 + z_1^2 - 27S(y_1 + 3G^2))^2 + 81S(y_1^2 + 2Gz_1 - 3S)^2}}{2}}$$

$$\theta = \arccos \frac{y_2}{\sqrt{y_2^2 + z_2^2}} \quad y_3 = \frac{-4G + \sqrt[6]{y_2^2 + z_2^2} \cos \frac{\theta}{3}}{3} - \frac{y_1 \cos \frac{\theta}{3} + 3 \operatorname{sgn}(z_2) \sqrt{S} \sin \frac{\theta}{3}}{3 \sqrt[6]{y_2^2 + z_2^2}}$$

$$z_3 = \frac{\operatorname{sgn}(z_2) \sqrt[6]{y_2^2 + z_2^2} \sin \frac{\theta}{3}}{3} - \frac{3\sqrt{S} \cos \frac{\theta}{3} - \operatorname{sgn}(z_2) y_1 \sin \frac{\theta}{3}}{3 \sqrt[6]{y_2^2 + z_2^2}}$$

$$y_4 = \frac{H \sqrt{y_3 + 2G + \sqrt{(y_3 + 2G)^2 + z_3^2}}}{\sqrt{(y_3 + 2G)^2 + z_3^2}} \quad z_4 = \frac{\operatorname{sgn}(z_3) H \sqrt{-y_3 - 2G + \sqrt{(y_3 + 2G)^2 + z_3^2}}}{\sqrt{(y_3 + 2G)^2 + z_3^2}}$$

公式 20:  $x_{1,2} = \frac{-2b - \sqrt{y_3 + 2G + \sqrt{(y_3 + 2G)^2 + z_3^2}} + \sqrt{-y_3 + 4y_4 + \sqrt{(y_3 - 4y_4)^2 + (z_3 + 4z_4)^2}}}{16a}$

$$\pm \frac{\operatorname{sgn}(z_3) \sqrt{-y_3 - 2G + \sqrt{(y_3 + 2G)^2 + z_3^2}} + \operatorname{sgn}(z_3 + 4z_4) \sqrt{y_3 - 4y_4 + \sqrt{(y_3 - 4y_4)^2 + (z_3 + 4z_4)^2}}}{16a} i$$

$$x_{3,4} = \frac{-2b - \sqrt{y_3 + 2G + \sqrt{(y_3 + 2G)^2 + z_3^2}} - \sqrt{-y_3 + 4y_4 + \sqrt{(y_3 - 4y_4)^2 + (z_3 + 4z_4)^2}}}{16a}$$

$$\pm \frac{\operatorname{sgn}(z_3) \sqrt{-y_3 - 2G + \sqrt{(y_3 + 2G)^2 + z_3^2}} - \operatorname{sgn}(z_3 + 4z_4) \sqrt{y_3 - 4y_4 + \sqrt{(y_3 - 4y_4)^2 + (z_3 + 4z_4)^2}}}{16a} i$$

$$x_{5,6} = \frac{-2b + \sqrt{y_3 + 2G + \sqrt{(y_3 + 2G)^2 + z_3^2}} + \sqrt{-y_3 - 4y_4 + \sqrt{(y_3 + 4y_4)^2 + (z_3 - 4z_4)^2}}}{16a}$$

$$\pm \frac{\operatorname{sgn}(z_3) \sqrt{-y_3 - 2G + \sqrt{(y_3 + 2G)^2 + z_3^2}} - \operatorname{sgn}(z_3 - 4z_4) \sqrt{y_3 + 4y_4 + \sqrt{(y_3 + 4y_4)^2 + (z_3 - 4z_4)^2}}}{16a} i$$

$$x_{7,8} = \frac{-2b + \sqrt{y_3 + 2G + \sqrt{(y_3 + 2G)^2 + z_3^2}} - \sqrt{-y_3 - 4y_4 + \sqrt{(y_3 + 4y_4)^2 + (z_3 - 4z_4)^2}}}{16a}$$

$$\pm \frac{\operatorname{sgn}(z_3) \sqrt{-y_3 - 2G + \sqrt{(y_3 + 2G)^2 + z_3^2}} + \operatorname{sgn}(z_3 - 4z_4) \sqrt{y_3 + 4y_4 + \sqrt{(y_3 + 4y_4)^2 + (z_3 - 4z_4)^2}}}{16a} i$$