

一元十次方程求根公式 (天珩公式)

標準型：(係數 $\in\mathbb{R}$ 且 $a\neq 0$)

$$ax^{10}+bx^9+cx^8+dx^7+ex^6+fx^5+gx^4+hx^3+jx^2+kx+l=0$$

重根判別式：

$$\begin{cases} E=9b^2-20ac \\ F=6b^3-20abc+25a^2d \\ G=28b^2c-175abd+400a^2e \\ H=14b^3c-105ab^2d+375a^2be-625a^3f \\ J=28b^4c-245ab^3d+1200a^2b^2e-4000a^3bf+8000a^4g \\ K=2b^5c-20ab^4d+125a^2b^3e-625a^3b^2f+2500a^4bg-6250a^5h \\ L=7b^5d-130ab^4e+1450a^2b^3f-12200a^3b^2g+78500a^4bh-320000a^5j \\ M=b^6d-20ab^5e+250a^2b^4f-2500a^3b^3g+21250a^4b^2h-150000a^5bj+750000a^6k \\ N=3b^6e-100ab^5f+2050a^2b^4g-33625a^3b^3h+485000a^4b^2j-6425000a^5bk+80000000a^6l \end{cases}$$

$$\begin{cases} A=2E^2-5aG \\ B=4E^3-15aEG-25aJ \\ C=14E^4+50a^2G^2-70aE^2G-100aEJ+125a^2L \\ D=-8E^5+50aE^3G+250aE^2J-625a^2EL+250a^3N \end{cases}$$

$$(*)\{P=8B^4+C^3+2BCD$$

總判別式： $\Delta=D^2-16A^5$

①當 $E=F=G=H=J=K=L=M=N=0$ 時，方程有一個十重實根。

$$\text{公式 1: } x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = x_7 = x_8 = x_9 = x_{10} = -\frac{b}{10a} = -\frac{2c}{9b} = -\frac{3d}{8c} = -\frac{4e}{7d} = -\frac{5f}{6e} = -\frac{6g}{5f} = -\frac{7h}{4g} = -\frac{8j}{3h} = -\frac{9k}{2j} = -\frac{10l}{k}$$

②當 $F=H=K=M=A=B=C=D=0$ 時，方程有兩個五重根。

(1)若 $9b^2 > 20ac$ ，方程有十個實根。

$$\text{公式 2: } x_{1,2} = x_{3,4} = x_{5,6} = x_{7,8} = x_{9,10} = \frac{-b \pm \sqrt{9b^2 - 20ac}}{10a}$$

(2)若 $9b^2 < 20ac$ ，方程有十個虛根。

$$\text{公式 3: } x_{1,2} = x_{3,4} = x_{5,6} = x_{7,8} = x_{9,10} = -\frac{b}{10a} \pm \frac{\sqrt{20ac - 9b^2}}{10a}i$$

③當 $AD \neq 0$ ， $F=H=K=M=B=C=\Delta=0$ 時，方程有四個二重根。

$$\text{公式 4: } x_{1,2} = \frac{-2Ab \pm \sqrt{4A^2E - 2D}}{20aA}$$

$$x_{3,4} = x_{5,6} = \frac{-4Ab \pm \sqrt{2D + 8A^2(2E + \sqrt{5A})}}{40aA}$$

$$x_{7,8} = x_{9,10} = \frac{-4Ab \pm \sqrt{2D + 8A^2(2E - \sqrt{5A})}}{40aA}$$

④當 $\Delta > 0$ ， $F=H=K=M=B=C=0$ 時，方程有至少八個虛根。

$$z_{1,2} = \frac{D \pm \sqrt{D^2 - 16A^5}}{4} \quad y_{1,2} = E^2 - \frac{-1 \pm \sqrt{5}}{2} E(\sqrt[5]{z_1} + \sqrt[5]{z_2}) + (\sqrt[5]{z_1} + \sqrt[5]{z_2})^2 - \frac{5 \pm \sqrt{5}}{2} A$$

$$\text{公式 5: } x_{1,2} = \frac{-b \pm \sqrt{E - (\sqrt[5]{z_1} + \sqrt[5]{z_2})}}{10a}$$

$$x_{3,4} = \frac{-2b + \sqrt{2E + (1 - \sqrt{5})(\sqrt[5]{z_1} + \sqrt[5]{z_2})/2 + 2\sqrt{y_1}}}{20a} \pm \frac{\sqrt{-2E - (1 - \sqrt{5})(\sqrt[5]{z_1} + \sqrt[5]{z_2})/2 + 2\sqrt{y_1}}}{20a} i$$

$$x_{5,6} = \frac{-2b - \sqrt{2E + (1 - \sqrt{5})(\sqrt[5]{z_1} + \sqrt[5]{z_2})/2 + 2\sqrt{y_1}}}{20a} \pm \frac{\sqrt{-2E - (1 - \sqrt{5})(\sqrt[5]{z_1} + \sqrt[5]{z_2})/2 + 2\sqrt{y_1}}}{20a} i$$

$$x_{7,8} = \frac{-2b + \sqrt{2E + (1 + \sqrt{5})(\sqrt[5]{z_1} + \sqrt[5]{z_2})/2 + 2\sqrt{y_2}}}{20a} \pm \frac{\sqrt{-2E - (1 + \sqrt{5})(\sqrt[5]{z_1} + \sqrt[5]{z_2})/2 + 2\sqrt{y_2}}}{20a} i$$

$$x_{9,10} = \frac{-2b - \sqrt{2E + (1 + \sqrt{5})(\sqrt[5]{z_1} + \sqrt[5]{z_2})/2 + 2\sqrt{y_2}}}{20a} \pm \frac{\sqrt{-2E - (1 + \sqrt{5})(\sqrt[5]{z_1} + \sqrt[5]{z_2})/2 + 2\sqrt{y_2}}}{20a} i$$

⑤當 $\Delta < 0$, $F=H=K=M=B=C=0$ 時。

$$\theta = \arccos \frac{D}{4A^2\sqrt{A}}$$

$$\text{公式 6: } x_{1,2} = \frac{-b \pm \sqrt{E - 2\sqrt{A} \cos \frac{\theta}{5}}}{10a}$$

$$x_{3,4} = \frac{-b \pm \sqrt{E + \sqrt{A} \left(\frac{1 - \sqrt{5}}{2} \cos \frac{\theta}{5} + \frac{\sqrt{10 + 2\sqrt{5}}}{2} \sin \frac{\theta}{5} \right)}}{10a} \quad x_{5,6} = \frac{-b \pm \sqrt{E + \sqrt{A} \left(\frac{1 - \sqrt{5}}{2} \cos \frac{\theta}{5} - \frac{\sqrt{10 + 2\sqrt{5}}}{2} \sin \frac{\theta}{5} \right)}}{10a}$$

$$x_{7,8} = \frac{-b \pm \sqrt{E + \sqrt{A} \left(\frac{1 + \sqrt{5}}{2} \cos \frac{\theta}{5} + \frac{\sqrt{10 - 2\sqrt{5}}}{2} \sin \frac{\theta}{5} \right)}}{10a} \quad x_{9,10} = \frac{-b \pm \sqrt{E + \sqrt{A} \left(\frac{1 + \sqrt{5}}{2} \cos \frac{\theta}{5} - \frac{\sqrt{10 - 2\sqrt{5}}}{2} \sin \frac{\theta}{5} \right)}}{10a}$$

⑥當 $H \neq 0$, $E=F=G=J=K=L=M=0$ 時。

(1)若 $256H^2 \geq 5aN$, 方程有兩個實根和八個虛根。

$$\text{公式 7: } x_{1,2} = \frac{-b \pm \sqrt[5]{5a(16H \pm \sqrt{256H^2 - 5aN})}}{10a}$$

$$x_{3,4} = \frac{-4b - (1 + \sqrt{5}) \sqrt[5]{5a(16H + \sqrt{256H^2 - 5aN})}}{40a} \pm \frac{\sqrt{10 - 2\sqrt{5}} \sqrt[5]{5a(16H + \sqrt{256H^2 - 5aN})}}{40a} i$$

$$x_{5,6} = \frac{-4b - (1 + \sqrt{5}) \sqrt[5]{5a(16H - \sqrt{256H^2 - 5aN})}}{40a} \pm \frac{\sqrt{10 - 2\sqrt{5}} \sqrt[5]{5a(16H - \sqrt{256H^2 - 5aN})}}{40a} i$$

$$x_{7,8} = \frac{-4b - (1 - \sqrt{5}) \sqrt[5]{5a(16H + \sqrt{256H^2 - 5aN})}}{40a} \pm \frac{\sqrt{10 + 2\sqrt{5}} \sqrt[5]{5a(16H + \sqrt{256H^2 - 5aN})}}{40a} i$$

$$x_{9,10} = \frac{-4b - (1 - \sqrt{5}) \sqrt[5]{5a(16H - \sqrt{256H^2 - 5aN})}}{40a} \pm \frac{\sqrt{10 + 2\sqrt{5}} \sqrt[5]{5a(16H - \sqrt{256H^2 - 5aN})}}{40a} i$$

(2)若 $256H^2 < 5aN$, 方程有十個虛根。

$$\theta = \arccos \frac{16H}{\sqrt{5aN}}$$

$$\text{公式 8: } x_{1,2} = \frac{-b + \sqrt[5]{5a^{10}\sqrt{5aN} \cos \frac{\theta}{5}}}{10a} \pm \frac{\sqrt[5]{5a^{10}\sqrt{5aN} \sin \frac{\theta}{5}}}{10a} i$$

$$x_{3,4} = \frac{-b + \sqrt[5]{5a^{10}\sqrt{5aN} \cos \frac{\theta + 2\pi}{5}}}{10a} \pm \frac{\sqrt[5]{5a^{10}\sqrt{5aN} \sin \frac{\theta + 2\pi}{5}}}{10a} i$$

$$X_{5,6} = \frac{-b + \sqrt[5]{5a^{10}\sqrt{5a}N} \cos \frac{\theta-2\pi}{5}}{10a} \pm \frac{\sqrt[5]{5a^{10}\sqrt{5a}N} \sin \frac{\theta-2\pi}{5}}{10a} i$$

$$X_{7,8} = \frac{-b + \sqrt[5]{5a^{10}\sqrt{5a}N} \cos \frac{\theta+4\pi}{5}}{10a} \pm \frac{\sqrt[5]{5a^{10}\sqrt{5a}N} \sin \frac{\theta+4\pi}{5}}{10a} i$$

$$X_{9,10} = \frac{-b + \sqrt[5]{5a^{10}\sqrt{5a}N} \cos \frac{\theta-4\pi}{5}}{10a} \pm \frac{\sqrt[5]{5a^{10}\sqrt{5a}N} \sin \frac{\theta-4\pi}{5}}{10a} i$$

⑦當 $BC \neq 0$, $F=H=K=M=A=P=0$ 時, 方程有至少八個虛根。

$$y_{1,2} = E^2 - E \left(\frac{1 \pm \sqrt{5}}{2} \sqrt[5]{\frac{C^2}{4B}} + \frac{1 \mp \sqrt{5}}{2} \sqrt[5]{\frac{2B^3}{C}} \right) + \left(\sqrt[5]{\frac{C^2}{4B}} + \sqrt[5]{\frac{2B^3}{C}} \right)^2 - \frac{5 \pm \sqrt{5}}{2} \sqrt[5]{\frac{B^2C}{2}}$$

$$\text{公式 9: } x_{1,2} = \frac{-b \pm \sqrt{E + \sqrt[5]{\frac{C^2}{4B}} + \sqrt[5]{\frac{2B^3}{C}}}}{10a}$$

$$X_{3,4} = \frac{-2b + \sqrt{2E - \frac{1+\sqrt{5}}{2} \sqrt[5]{\frac{C^2}{4B}} - \frac{1-\sqrt{5}}{2} \sqrt[5]{\frac{2B^3}{C}} + 2\sqrt{y_1}}}{20a} \pm \frac{\sqrt{-2E + \frac{1+\sqrt{5}}{2} \sqrt[5]{\frac{C^2}{4B}} + \frac{1-\sqrt{5}}{2} \sqrt[5]{\frac{2B^3}{C}} + 2\sqrt{y_1}}}{20a} i$$

$$X_{5,6} = \frac{-2b - \sqrt{2E - \frac{1+\sqrt{5}}{2} \sqrt[5]{\frac{C^2}{4B}} - \frac{1-\sqrt{5}}{2} \sqrt[5]{\frac{2B^3}{C}} + 2\sqrt{y_1}}}{20a} \pm \frac{\sqrt{-2E + \frac{1+\sqrt{5}}{2} \sqrt[5]{\frac{C^2}{4B}} + \frac{1-\sqrt{5}}{2} \sqrt[5]{\frac{2B^3}{C}} + 2\sqrt{y_1}}}{20a} i$$

$$X_{7,8} = \frac{-2b + \sqrt{2E - \frac{1-\sqrt{5}}{2} \sqrt[5]{\frac{C^2}{4B}} - \frac{1+\sqrt{5}}{2} \sqrt[5]{\frac{2B^3}{C}} + 2\sqrt{y_2}}}{20a} \pm \frac{\sqrt{-2E + \frac{1-\sqrt{5}}{2} \sqrt[5]{\frac{C^2}{4B}} + \frac{1+\sqrt{5}}{2} \sqrt[5]{\frac{2B^3}{C}} + 2\sqrt{y_2}}}{20a} i$$

$$X_{9,10} = \frac{-2b - \sqrt{2E - \frac{1-\sqrt{5}}{2} \sqrt[5]{\frac{C^2}{4B}} - \frac{1+\sqrt{5}}{2} \sqrt[5]{\frac{2B^3}{C}} + 2\sqrt{y_2}}}{20a} \pm \frac{\sqrt{-2E + \frac{1-\sqrt{5}}{2} \sqrt[5]{\frac{C^2}{4B}} + \frac{1+\sqrt{5}}{2} \sqrt[5]{\frac{2B^3}{C}} + 2\sqrt{y_2}}}{20a} i$$